MATHEMATICS
MODERN ALGEBRA

Previous year Questions from 1992 To 2017

Syllabus

Groups, subgroups, cyclic groups, cosets, Lagrange’s Theorem, normal subgroups, quotient groups, homomorphism of groups, basic isomorphism theorems, permutation groups, Cayley’s theorem.

Rings, subrings and ideals, homomorphisms of rings; Integral domains, principal ideal domains, Euclidean domains and unique factorization domains; Fields, quotient fields.

**Note: Syllabus was revised in 1990’s and 2001 & 2008**
2017
1. Let $F$ be a field and $F[X]$ denote the ring of polynomials over $F$ in a single variable $X$. For $f(X), g(X) \in F[X]$ with $g(X) \neq 0$, show that there exist $q(X), r(X) \in F[X]$ such that $f(X) = q(X)g(X) + r(X)$ and $\deg(r(X)) < \deg(g(X))$. $(20 \text{ marks})$

2. Show that the groups $\mathbb{Z}_2 \times \mathbb{Z}_7$ and $\mathbb{Z}_{35}$ are isomorphic. $(15 \text{ marks})$

2016
3. Let $K$ be a field and $K[X]$ be the ring of polynomials over $K$ in a single variable $X$ for a polynomial $f \in K[X]$ Let $(f)$ denote the ideal in $K[X]$ generated by $f$. Show that $(f)$ is a maximal ideal in $K[X]$ if and only if $f$ is an irreducible polynomial over $K$. $(10 \text{ marks})$

4. Let $p$ be prime number and $\mathbb{Z}_p$ denote the additive group of integers modulo $p$. Show that every non-zero element $\mathbb{Z}_p$ generates $\mathbb{Z}_p$. $(15 \text{ marks})$

5. Let $K$ be an extension of field $F$ prove that the element of $K$ which are algebraic over $F$ form a subfield of $K$. Further if $F \subseteq K \subseteq L$ are fields $L$ is algebraic over $K$ and $K$ is algebraic over $F$ then prove that $L$ is algebraic over $F$. $(20 \text{ marks})$

6. Show that every algebraically closed field is infinite. $(15 \text{ marks})$

2015
7. (i) How many generators are there of the cyclic group $G$ of order 8? Explain. $(5 \text{ marks})$

(ii) Taking a group $\{e, a, b, c\}$ of order 4, where $e$ is the identity, construct composition tables showing that one is cyclic while the other is not. $(5 \text{ marks})$

8. Give an example of a ring having identity but a subring of this having a different identity. $(10 \text{ marks})$

9. If $R$ is a ring with unit element 1 and $\phi$ is a homomorphism of $R$ onto $R'$, prove that $\phi(1)$ is the unit element of $R'$. $(15 \text{ marks})$

10. Do the following sets form integral domains with respect to ordinary addition and multiplication? Is so, state if they are fields:
    (i) The set of numbers of the form $b\sqrt{2}$ with $b$ rational
    (ii) The set of even integers.
    (iii) The set of positive integers. $(5+6+4=15 \text{ marks})$
11. Let $G$ be the set of all real $2 \times 2 \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where $xz \neq 0$ matrices where. Show that $G$ is group under matrix multiplication. Let $N$ denote the subset $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in R \right\}$. Is $N$ a normal subgroup of $G$? Justify your answer. (10 marks)

12. Show that $Z_7$ is a field. Then find $(5+6)^{-1}$ and $(-4)^{-1}$ in $Z_7$. (15 marks)

13. Show that the set $\{a + b\omega : \omega^3 = 1\}$, where $a$ and $b$ are real numbers, is a field with respect to usual addition and multiplication. (15 marks)

14. Prove that the set $Q(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in Q\}$ is commutative ring with identity. (15 marks)

15. Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in R \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f: C \to S$ defined by $f(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that $f$ is an isomorphism. (Here $R$ is the set of real numbers and $C$ is the set of complex numbers)? (10 marks)

16. Give an example of an infinite group in which every element has finite order. (10 marks)

17. What are the orders of the following permutation in $S_{10}$? $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$ and $(1 \ 2 \ 3 \ 4 \ 5)(6 \ 7)$ (10 marks)

18. What is the maximal possible order of an element in $S_{10}$? Why? Give an example of such an element. How many elements will there be in $S_{10}$ of that order? (13 marks)

19. Let $J = \{a + ib / a, b \in Z\}$ be the ring of Gaussian integers (subring of $C$). Which of the following is $J$: Euclidean domain, principal ideal domain, and unique factorization domain? Justify your answer (15 marks)

20. Let $R^c =$ ring of all real value continuous function on $[0,1]$, under the operations $(f + g)x = f(x) + g(x), (fg)x = f(x)g(x)$. Let $M = \left\{ f \in R^c / f\left(\frac{1}{2}\right) = 0 \right\}$. Is $M$ a maximal ideal of $R$? Justify your answer. (15 marks)
2012

21. How many elements of order 2 are there in the group of order 16 generated by \( a \) and \( b \) such that the order of \( a \) is 8, the order of \( b \) is 2 and \( bab^{-1} = a^{-1} \). (12 marks)

22. How many conjugacy classes does the permutation groups \( S_5 \) of permutations 5 numbers have? Write down one element in each class (preferably in terms of cycles). (15 marks)

23. Is the ideal generated by 2 and \( X \) in the polynomial ring \( \mathbb{Z}[X] \) of polynomials in a single variable \( X \) with coefficients in the ring of integers \( \mathbb{Z} \), a principal ideal? Justify your answer. (15 marks)

24. Describe the maximal ideals in the ring of Gaussian integers \( \mathbb{Z}[i] = \{a + ib / a, b \in \mathbb{Z}\} \). (20 marks)

2011

25. Show that the set \( G = \{f_1, f_2, f_3, f_4, f_5, f_6\} \) of six transformations on the set of Complex numbers defined by

\[
\begin{align*}
    f_1(z) &= z, \\
    f_2(z) &= 1 - z, \\
    f_3(z) &= \frac{z}{1 - z}, \\
    f_4(z) &= \frac{1}{z}, \\
    f_5(z) &= \frac{1}{1 - z}, \\
    f_6(z) &= \frac{(z - 1)}{z}
\end{align*}
\]

is a nonabelian group of order 6 w.r.t. composition of mappings (12 marks)

26. Prove that a group of Prime order is abelian. (6 marks)

27. How many generators are there of the cyclic group \((G, \cdot)\) of order 8? (6 marks)

28. Give an example of a group \( G \) in which every proper subgroup is cyclic but the group itself is not cyclic. (15 marks)

29. Let \( F \) be the set of all real valued continuous functions defined on the closed interval \([0,1]\). Prove that \((F, +, \cdot)\) is a Commutative Ring with unity with respect to addition and multiplication of functions defined point wise as below:

\[
\begin{align*}
    (f + g)(x) &= f(x) + g(x) \quad \text{where } f, g \in F \\
    (fg)(x) &= f(x)g(x)
\end{align*}
\]

and \( x \in [0,1] \) (15 marks)

30. Let \( a \) and \( b \) be elements of a group, with \( a^2 = e \), \( b^6 = e \) and \( ab = b^4 a \). Find the order of \( ab \), and express its inverse in each of the forms \( a^m b^n \) and \( b^m a^n \). (20 marks)

2010

31. Let \( G = R - \{-1\} \) be the set of all real numbers omitting \(-1\). Define the binary relation \(*\) on \( G \) by \( a * b = a + b + ab \). Show \((G, *)\) is a group and it is abelian. (12 marks)

32. Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify. (12 marks)

33. Let \((R^*, \cdot)\) be the multiplicative group of non-zero reals and \((GL(n,R), X)\) be the multiplicative group of \( n \times n \) non-singular real matrices. Show that the quotient group \( \frac{GL(n,R)}{SL(n,R)} \) and \((R^*, \cdot)\) are isomorphic where \( SL(n,R) = \{ A \in GL(n,R) / \det A = 1 \} \). What is the center of \( GL(n,R) \)? (15 marks)
34. Let $C = \{ f : I = [0,1] \rightarrow R / f$ is continuous $\}$ Show $C$ is a commutative ring with 1 under point wise addition and multiplication. Determine whether $C$ is an integral domain. Explain.

35. Consider the polynomial ring $\mathbb{Q}[x]$. Show $p(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$. Let $I$ be the ideal $\mathbb{Q}[x]$ in generated by $p(x)$. Then show that $\frac{\mathbb{Q}[x]}{I}$ is field and that each element of it is of the form $a_0 + a_1 t + a_2 t^2$ with $a_0, a_1, a_2$ in $\mathbb{Q}$ and $t = x + 1$.

36. Show that the quotient ring $\frac{\mathbb{Z}[i]}{1+3i}$ is isomorphic to the ring $\frac{\mathbb{Z}}{10\mathbb{Z}}$ where $\mathbb{Z}[i]$ denotes the ring of Gaussian integers.

37. If $R$ is the set of real numbers and $R_+$ is the set of positive real numbers, show that $R$ under addition ($R, +$) and $R_+$ under multiplication ($R_+, \cdot$) are isomorphic. Similarly if $Q$ is set of rational numbers and $Q_+$ is the set of positive rational numbers, are $(Q, +)$ and $(Q_+, \cdot)$ isomorphic? Justify your answer.

38. Determine the number of homomorphisms from the additive group $\mathbb{Z}_{15}$ to the additive group $\mathbb{Z}_{10}$ ($\mathbb{Z}_n$ is the cyclic group of order $n$).

39. How many proper, non-zero ideals does the ring $\mathbb{Z}_{12}$ have? Justify your answer. How many ideals does the ring $\mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$ have? Why?

40. Show that the alternating group of four letters $A_4$ has no subgroup of order 6.

41. Show that $\mathbb{Z}[X]$ is a unique factorization domain that is not a principal ideal domain ($\mathbb{Z}$ is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain? ($\mathbb{Z}[X]$ is the ring of polynomial in the variable $X$ with integer.)

42. How many elements does the quotient ring $\frac{\mathbb{Z}[X]}{X^2 + 1}$ have? Is it an integral domain?

2009

43. Let $R_0$ be the set of all real numbers except zero. Define a binary operation $*$ on $R_0$ as $a * b = |a| |b| |a|$ where $|a|$ denotes absolute value of $a$. Does $(R_0, *)$ form a group? Examine.

44. Suppose that there is a positive even integer $n$ such that $a^n = a$ for all the elements $a$ of some ring $R$. Show that $a + a = 0$ for all $a \in R$ and $a + b = 0 \Rightarrow a = b$ for all $a, b \in R$.
45. Let $G$ and $\overline{G}$ be two groups and let $\phi : G \rightarrow \overline{G}$ be a homomorphism. For any element $a \in G$
(i) Prove that $O(\phi(a)) \mid O(a)$
(ii) Ker $\phi$ is normal subgroup of $G$. (15 marks)

46. Let $R$ be a ring with unity. If the product of any two non-zero elements is non-zero. Then prove that $ab = 1 \Rightarrow ba = 1$. Whether $Z_6$ has the above property or not explain. Is $Z_6$ an integral domain? (15 marks)

47. Prove that every integral domain can be embedded in a field. (15 marks)

48. Show that any maximal ideal in the commutative ring $F[x]$ of polynomial over a field $F$ is the principal ideal generated by an irreducible polynomial. (15 marks)

2007

49. If in a group $G$, $a^5 = e$, $e$ is the identity element of $G$, then find the order of $b$. (12 marks)

50. Let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a,b,c,d \in \mathbb{Z}$. Show that $R$ is a ring under matrix addition and multiplication $\left\{ A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, a,b \in \mathbb{Z} \right\}$. Then show that $A$ is a left ideal of $R$ but not a right ideal of $R$. (12 marks)

51. (i) Prove that there exists no simple group of order 48.
(ii) $1 + \sqrt{-3}$ and $Z[\sqrt{-3}]$ is an irreducible element, but not prime. Justify your answer. (15 marks)

52. Show that in the ring $R = \left\{ a + b\sqrt{-5} \mid a,b \in \mathbb{Z} \right\}$, The element $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but $\alpha\gamma$ and $\beta\gamma$ have no g.c.d in $R$, where $\gamma = 7(1 + 2\sqrt{-5})$. (30 marks)

2006

53. Let $S$ be the set of all real numbers except –1. Define on $S$ by $a * b = a + b + ab$. Is $(S, *)$ a group? Find the solution of the equation $2 * x * 3 = 7$ in $S$. (12 marks)

54. If $G$ is a group of real numbers under addition and $N$ is the subgroup of $G$ consisting of integers, Prove that $\frac{G}{N}$ is isomorphic to the group $H$ of all complex numbers of absolute value 1 under multiplication (12 marks)

55. (i) Let $O(G) = 108$. Show that there exists a normal subgroup or order 27 or 9.
(ii) Let $G$ be the set of all those ordered paris $(a,b)$ real numbers for which $a \neq 0$ and define in $G$, an operation as follows: $(a,b) \otimes (c,d) = (ac, bc + d)$ Examine whether $G$ is a group w.r.t the operation $\otimes$. If it is a group, is $G$ abelian? (10 marks)

56. Show that $Z[\sqrt{2}] = \left\{ a + b\sqrt{2} : a,b \in \mathbb{Z} \right\}$ is a Euclidean domain. (30 marks)
57. If \( M \) and \( N \) are normal subgroups of a group \( G \) such that \( M \cap N = \{e\} \), show that every element of \( M \) commutes with every element of \( N \). \( (12 \text{ marks}) \)

58. Show that \((1+i)\) is a prime element in the ring \( R \) of Gaussian integers. \( (12 \text{ marks}) \)

59. Let \( H \) and \( K \) be two subgroups of a finite group \( G \) such that \( |H| > \sqrt{|G|} \) and \( |K| > \sqrt{|G|} \).

Prove that \( H \cap K \neq \{e\} \). \( (15 \text{ marks}) \)

60. If \( f : G \to G' \) is an isomorphism, prove that the order \( a \in G \) of is equal to the order of \( f(a) \)

61. Prove that any polynomial ring \( F[x] \) over a field \( F \) is U.F.D \( (30 \text{ marks}) \)

62. If \( p \) is prime number of the form \( 4n+1 \), \( n \) being a natural number, then show that congruence \( x^2 \equiv -1 \) (mod \( p \)) is solvable. \( (12 \text{ marks}) \)

63. Let \( G \) be group such that of all \( a, b \in G \) (i) \( ab = ba \) (ii) \( O(a), O(b) \) = 1 then show that \( O(ab) = O(a) \cdot O(b) \)

64. Verify that the set \( E \) of the four roots of \( x^4 - 1 = 0 \) forms a multiplicative group. Also prove that a transformation \( T, T(n)=i^n \) is a homomorphism from \( I \), (Group of all integers with addition) onto \( E \) under multiplication. \( (10 \text{ marks}) \)

65. Prove that if cancellation law holds for a ring \( R \) then \( a(\neq 0) \in R \) is not a zero divisor and conversely \( (10 \text{ marks}) \)

66. The residue class ring \( \mathbb{Z}/(m) \) is a field iff \( m \) is a prime integer. \( (15 \text{ marks}) \)

67. Define irreducible element and prime element in an integral domain \( D \) with units. Prove that every prime element in \( D \) is irreducible and converse of this is not (in general) true. \( (25 \text{ marks}) \)

2004

68. If \( H \) is a subgroup of a group \( G \) such that \( x^2 \in H \) for every \( x \in G \), then prove that \( H \) is a normal subgroup of \( G \). \( (12 \text{ marks}) \)

69. Show that the ring \( \mathbb{Z}[i] = \{a + ib / a, b \in \mathbb{Z}, i = \sqrt{-1}\} \) of Gaussian integers is a Euclidean domain \( \quad (12 \text{ marks}) \)

70. Let \( R \) be the ring of all real-valued continuous functions on the closed interval \([0,1]\).

Let \( M = \left\{ f(x) \in R / f\left(\frac{1}{3}\right) = 0 \right\} \). Show that \( M \) is a maximal ideal of \( R \). \( (10 \text{ marks}) \)

71. Let \( M \) and \( N \) be two ideals of a ring \( R \). Show that \( M \cup N \) is an ideal of \( R \) if and only if either \( M \subseteq N \) or \( N \subseteq M \) \( (10 \text{ marks}) \)
72. Show that \(\mathbb{Q}(\sqrt{3}, i)\) is a splitting field for \(x^5 - 3x^3 + x^2 - 3\) where \(\mathbb{Q}\) is the field of rational numbers. (15 marks)

73. Prove that \(x^2 + x + 4\) is irreducible over \(F\) the field to integers modulo 11 and prove further that \(F[x]/(x^2 + x + 4)\) is a field having 121 elements. (15 marks)

74. Let \(R\) be a unique factorization domain (U.F.D), then prove that \(R[x]\) is also U.F.D. (10 marks)

2002

75. Show that a group of order 35 is cyclic. (12 marks)

76. Show that polynomial \(25x^4 + 9x^3 + 3x + 3\) is irreducible over the field of rational numbers. (12 marks)

77. Show that a group of \(p^2\) is abelian, where \(p\) is a prime number. (10 marks)

78. Prove that a group of order 42 has a normal subgroup of order 7. (10 marks)

79. Prove that in the ring \(F[x]\) of polynomial over a field \(F\), the ideal \(I = \langle p(x) \rangle\) is maximal if and only if the polynomial \(p(x)\) is irreducible over \(F\). (20 marks)

80. Show that every finite integral domain is a field. (10 marks)

81. Let \(F\) be a field with \(q\) elements. Let \(E\) be a finite extension of degree \(n\) over \(F\). Show that \(E\) has \(q^n\) elements. (10 marks)

2001

82. Let \(K\) be a field and \(G\) be a finite subgroup of the multiplicative group of non-zero elements of \(K\). Show that \(G\) is a cyclic group. (12 marks)

83. Let \(n\) be a fixed positive integer and \(\mathbb{Z}_n\) be the ring of integers modulo \(n\). Let \(\{a \in \mathbb{Z}_n | a \neq 0\}\) and \(a\) is relatively prime to \(n\). Show that \(\mathbb{Z}_n\) is a group under multiplication defined in \(\mathbb{Z}_n\). Hence, or otherwise, show that \(a^{\phi(n)} \equiv 1 \pmod{n}\) for all integers \(a\) relatively prime to \(n\) where \(\phi(n)\) denotes the number of positive integers that are less than \(n\) and are relatively prime to \(n\). (20 marks)

2000

84. If \(R\) is a commutative ring with unit element and \(M\) is an ideal of \(R\), then show that \(R/M\) is a field if and only if \(M\) is a maximal ideal of \(R\). (20 marks)

85. Prove that every finite extension of a field is an algebraic extension. Give an example to show that the converse is not true. (20 marks)
88. Let $M$ be a subgroup and $N$ a normal subgroup of $G$. Show that $MN$ is a subgroup of $G$ and $\frac{MN}{N}$ is isomorphic to $\frac{M}{M \cap N}$. (20 marks)

89. Let $F$ be a finite field. Show that the characteristic of $F$ must be a prime integer $p$ and the number of elements in $F$ must be $p^m$ for some positive integer $m$. (20 marks)

90. Let $F$ be a field and $F[x]$ denote the set of all polynomials defined over $F$. If $f(x)$ is an irreducible polynomial in $F[x]$, show that the ideal generated by $f(x)$ in $F[x]$ is maximal and $\frac{F[x]}{f(x)}$ is a field. (20 marks)

91. Show that any finite commutative ring with no zero divisors must be a field. (20 marks)

1999

92. If $\phi$ is a homomorphism of $G$ into $G$ with kernel $K$, then show that $K$ is a normal subgroup of $G$. (20 marks)

93. If $p$ is prime number and $p^a \mid O(G)$, then prove that $G$ has a subgroup of order $p^a$. (20 marks)

94. Let $R$ be a commutative ring with unit element whose only ideals are $(0)$ and $R$ itself. Show that $R$ is a field. (20 marks)

1998

95. Prove that if a group has only four elements then it must be abelian. (20 marks)

96. If $H$ and $K$ are subgroups of a group $G$ then show that $HK$ is a subgroup of $G$ if and only if $HK=KH$. (20 marks)

97. Let $(R, +, \cdot)$ be a system satisfying all the axioms for a ring with unity with the possible exception of $a+b=b+a$. Prove that $(R, +, \cdot)$ is a ring. (20 marks)

98. If $p$ is prime then prove that $\mathbb{Z}_p$ is a field. Discuss the case when $p$ is not a prime number. (20 marks)

99. Let $D$ be a principal domain. Show that every element that is neither zero nor a unit in $D$ is a product of irreducible elements. (20 marks)

1997

100. Show that a necessary and sufficient condition for a subset $H$ of a group $G$ to be a subgroup is $HH^{-1}=H$. (20 marks)

101. Show that the order of each subgroup of a finite group is a divisor of the order of the group. (20 marks)

102. In a group $G$, the commutator $(a,b)$, $a, b \in G$ is the element $aba^{-1}b^{-1}$ and the smallest subgroup containing all commutators is called the commutator subgroup of $G$. Show that a quotient group $\frac{G}{H}$ is abelian if and only if $H$ contains the commutator subgroup of $G$. (20 marks)

103. If $x^2=x$ for all $x$ in a ring $R$, show that $R$ is commutative. Give an example to show that the converse is not true. (20 marks)
104. Show that an ideal $S$ of the ring of integers $\mathbb{Z}$ is maximal ideal if and only if $S$ is generated by a prime integer. \hspace{1cm} (20 \text{ marks})

105. Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true. \hspace{1cm} (20 \text{ marks})

1996

106. Let $R$ be the set of real numbers and $G = \{(a,b) \mid a, b \in R, a \neq 0\}. G \times G \rightarrow G$ is defined by $(a,b)*(c,d) = (ac,bc + d).$ Show that $(G, *)$ is a group. Is it abelian? Is $(H, *)$ a Subgroup of $(G, *)$ when $H = \{(1,b) \mid b \in R\}? \hspace{1cm} (20 \text{ marks})

107. Let $f$ be a homomorphism of a group $G$ onto a group $G'$ with kernel $H$. For each subgroup $K'$ of $G$ define $K$ by $k = \{x \in G \mid f(x) \in K'\}$. Prove that

(i) $K'$ is isomorphic to $\frac{K}{H}$

(ii) $\frac{G}{K}$ is isomorphic to $\frac{G'}{K'}$ \hspace{1cm} (20 \text{ marks})

108. Prove that a normal subgroup $H$ of a group $G$ is maximal, if and only if the quotient group $\frac{G}{H}$ is simple. \hspace{1cm} (20 \text{ marks})

109. In a ring $R$, prove that cancellation laws hold if and only if $R$ has no zero divisors. \hspace{1cm} (20 \text{ marks})

110. If $S$ is an ideal of a ring $R$ and $T$ any subring of $R$, then prove that $S$ is an ideal of $S + T = \{s + t \mid s \in S, t \in T\}$. \hspace{1cm} (20 \text{ marks})

111. Prove that the polynomial $x^2 + x + 4$ is irreducible over the field of integers modulo 11. \hspace{1cm} (20 \text{ marks})

1995

112. Let $G$ be a finite set closed under an associative binary operation such that $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$ for all $a, b, c \in G$. Prove that $G$ is a group. \hspace{1cm} (20 \text{ marks})

113. Let $G$ be group of order $p^n$. Where $p$ is a prime number and $n > 0$. Let $H$ be a proper subgroup of $G$ and $N(H) = \{x \in G : x^{-1}hx \in H \forall h \in H\}$. Prove that $N(H) \neq H$ \hspace{1cm} (20 \text{ marks})

114. Show that a group of order 112 is not simple \hspace{1cm} (20 \text{ marks})

115. Let $R$ be a ring with identity. Suppose there is an element $a$ of $R$ which has more than one right inverse. Prove that $a$ has infinitely many right inverses. \hspace{1cm} (20 \text{ marks})

116. Let $F$ be a field and let $p(x)$ be an irreducible polynomial over $F$. Let $\langle p(x) \rangle$ be the ideal generated by $p(x)$. Prove that $\langle p(x) \rangle$ is a maximal ideal. \hspace{1cm} (20 \text{ marks})
117. Let $F$ be a field of characteristic $p \neq 0$. Let $F(x)$ be the polynomial ring. Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is an element of $F(x)$. Define $f(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1}$. If $f(x) = 0$, prove that there exist $g(x) \in F(x)$ such that $f(x) = g(x^p)$.

1994

118. If $G$ is a group such that $(ab)^n = a^n b^n$ for three consecutive integers $n$ for all $a, b \in G$, then prove that $G$ is abelian.

1993

119. Can a group of order 42 be simple? Justify your claim.

120. If $R$ is an integral domain (not necessarily a unique factorization domain) and $F$ is its field of quotients, then show that any element $f(x)$ in $F(x)$ is of the form $f(x) = \frac{f_0(x)}{a}$ where $f_0(x) \in R[x], a \in R$.

121. If $R$ is an integral domain (not necessarily a unique factorization domain) and $F$ is its field of quotients, then show that any element $f(x)$ in $F(x)$ is of the form $f(x) = \frac{f_0(x)}{a}$ where $f_0(x) \in R[x], a \in R$.

1992

122. If $G$ is a cyclic group of order $n$ and $p$ divides $n$, then prove that there is a homomorphism of $G$ onto a cyclic group of order $p$. What is the Kernel of homomorphism?
133. If $R$ is unique factorization domain, then prove that any $f \in R[x]$ is an irreducible element of $R[x]$, if and only if either $f$ is an irreducible element of $R$ or $f$ is an irreducible polynomial in $R[x]$.  

134. Prove that $x^2+1$ and $x^2+x+4$ are irreducible over $F$, the field of integer modulo 11. Prove also that $\frac{F[x]}{\langle x^2+1 \rangle}$ and $\frac{F[x]}{\langle x^2+x+4 \rangle}$ are isomorphic fields each having 121 elements.  

135. Find the degree of splitting field $x^5 - 3x^3 + x^2 - 3$ over $\mathbb{Q}$, the field of rational numbers.