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MATHEMATICS

COMPLEX ANALYSIS

Previous year Questions from **1992 To 2017**

Syllabus

Analytic functions, Cauchy-Riemann equations, Cauchy's theorem, Cauchy's integral formula, power series representation of an analytic function, Taylor's series; Singularities; Laurent's series; Cauchy's residue theorem; Contour integration.

**** Note: Syllabus was revised in 1990's and 2001 & 2008 ****

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2017

1. Using contour integral method, prove that $\int_0^\infty \frac{x\sin mx}{a^2+x^2}dx = \frac{\pi}{2}e^{-ma}$ **(15 Marks)**

2. Let f=u+iv be an analytic function on the unit disc $D = \{z \in C : |z| < 1\}$. Show that

   $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ at all points of D. **(15 Marks)**

3. For a function $f : C \to C$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)}$=f.

   Let f be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all k=,1,2,3,...

   Show that f is a polynomial. **(15 Marks)**

2016

4. Is $v(x,y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim, if yes find its conjugate harmonic function and hence obtain the analytic function u(x,y) whose real and imagi nary parts are *u* and *v* respectively. **(10 Marks)**

5. Let g; $[0,1] \to C$ be the curve $g(t) = e^{2pit}$, $0 \leq t \leq 1$ find giving justification the

   values of the contour integral $\int_\gamma \frac{dz}{4z^2 - 1}$ **(15 Marks)**

6. Prove that every power series represents an analytic function inside its circle of convergence. **(20 Marks)**

2015

7. Show that the function $v(x,y) = 1n\,(x^2+y^2)+x+y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z. **(10 Marks)**

8. Find all possible Taylor's and Laurent's series expansions of the function

   $f(z) = \frac{2z-3}{z^2 - 3z + 2}$ about the point z=0. **(20 Marks)**

9. State Cauchy's residue theorem. Using it, evaluate the intergral

   $\int_C \frac{e^z + 1}{z(z+1)(z-i)^2} dz; C : |z| = 2$. **(15 Marks)**

2014

10. Prove that the function *f(z)=u+iv*, where $f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z\neq 0; f(0)=0$ satis fies Cauchy-Riemann equations at the origin, but the derivative of *f* at *z=0* does not exist. **(10 Marks)**

11. Expand in Laurent series the function $f(z)=\frac{1}{z^2(z-1)}$ about $z=0$ and $z=1$.

(10 Marks)

12. Evaluate the integral $\int_0^{\pi}\frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues. **(20 Marks)**

2013

13. Prove that if $be^{a+1}<1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has *n* zeros in the unit circle. **(10 Marks)**

14. Using Cauchy's residue theorem, evaluate the integral $I=\int_0^{\pi}\sin^4\theta\, d\theta$.**(15 Marks)**

2012

15. Show that the function defined by $f(z)=\begin{cases}\frac{x^3y^5(x+xiy)}{x^6+y^{10}}, & z\neq 0\\ 0, & z\neq 0\end{cases}$ is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin.

(12 Marks)

16. Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4}dz$ Where *c* is the circle $|z|=2$.

(15 Marks)

17. Expand the function $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1<|z|<3$

(ii) $|z|>3$

(iii) $0<|z+1|<2$

(iv) $|z|<1$ **(15 Marks)**

18. Evaluate by contour integration $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}\ a^2 < 1$. **(15 Marks)**

2011

19. If *f(z)=u+iv* is an analytic function of z = x+iy and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find f(z) subject to the condition, $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$. **(12 Marks)**

20. If the function *f(z)* is analytic and one valued in $|z-a| < R$, Prove that for $0 < r < R$, $f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta) e^{-i\theta} d\theta$, Where P(θ) is the real part of $f(a+re^{i\theta})$

(15 Marks)

21. Evaluate by Contour integration, $\int_0^1 \frac{dx}{\left(x^2 - x^3\right)^{\frac{1}{3}}}$. **(15 Marks)**

22. Find the Laurent series for the function $f(z) = \frac{1}{1-z^2}$ with centre z=1.

(15 Marks)

2010

23. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of *u(x,y)*. Hence find the analytic function *f* for which *u(x,y)* is the real part. **(12 Marks)**

24. (i) Evaluate the line integral $\int_c f(z)dz$ where $f(z) = z^2$, *c* is the boundary of the triangle with vertices A(0,0), B(1,0), C(1,2) in that order.

(ii) Find the image of the finite vertical strip $R : x = 5$ to $x = 9, -\pi \le y \le \pi$ of *z*-plane under exponential function **(15 Marks)**

25. Find the Laurent series of the function

$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$ as $\sum_{n=-\infty}^{\infty} C_n z^n$ for 0, $|z| < \infty$ where

$C_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \lambda\sin\phi)d\phi, n = 0, \pm1, \pm2, \ldots$ with l a given complex number and taking the unit circle *C* given by $z = e^{i\phi} (-\pi \le \phi \le \pi)$ as contour in this region.

(15 Marks)

2009

26. Let $f(z)=\dfrac{a_0+a_1z+....+a_{n-1}z^{n-1}}{b_0+b_1z+......+b_nz^n}, b_n\neq 0.$ Assume that the zeros of the denominator are simple. show that the sum of the resdiues of $f(z)$ at its poles is equal to $\dfrac{a_n-1}{b_n}$ **(12 Marks)**

27. If α,β,γ are real numbers such that $\alpha^2>\beta^2+\gamma^2$ show that:

$$\int_0^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta}=\frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}$$ **(30 Marks)**

2008

28. Find the residue of $\dfrac{\cot z\coth z}{z^3}$ at z=0. **(12 Marks)**

29. Evaluate $\int_c\left[\dfrac{e^{2z}}{z^2\left(z^2+2z+2\right)}+\log\left(z-6\right)+\dfrac{1}{\left(z-4\right)^2}\right]dz$ where C is the circle $|z|=3$. State the theorems you use in evaluating above integral. **(15 Marks)**

2007

30. Prove that the function f defined by

$$f(z)=\begin{cases}\dfrac{z^5}{|z|^4} & z\neq 0\\ 0 & z=0\end{cases}$$ is not differentiable at z=0. **(12 Marks)**

31. Evaluate (by using residue therorem) $\int_0^{2\pi}\dfrac{d\theta}{1+8\cos^2\theta}$. **(15 Marks)**

32. Show that the trnasormation $w=z^2$ is conformal at point $z=1+i$ by finding the images of the lines $y=x$ and $x=1$ which intersect at $z=1+i$ **(15 Marks)**

2006

33. Determine all bilinear transformation which map the half plane $Im(z)\geq 0$ into the unit circle $|w|\leq 1$ **(12 Marks)**

34. With the aid of residues, evaluate $\int_0^{\pi}\dfrac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta,\ -1<a<1$ **(15 Marks)**

35. Prove that all the roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$ **(15 Marks)**

2005

36. If $f(z)=u+iv$ is an analytic function of the complex variable z and $u-v=e^x$ *(cosy–siny)*, determined *f(z)* in terms of *z*. **(12 Marks)**

37. Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for

    (i) $1<|z|<3$

    (ii) $|z|<3$ and

    (iii) $|z|<1$ **(30 Marks)**

2004

38. Find the image of the line *y=x* under the mapping $w=\dfrac{4}{z^2+1}$ and draw the same. Find the points where this transformation ceases to be conformal.**(12 Marks)**

39. If all zeros of a polynomial *P(z)* lies in a half plane then show that zeros of the derivatives *P'(z)* also lies in the same half plane. **(15 Marks)**

40. Using contour integration evaluate $\int_0^{2\pi}\dfrac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}d\theta,\ 0<p<1$

    (15 Marks)

2003

41. Determine all the bilinear transformations which transform the unit circle $|z|\le 1$ into the unit circle $|w|\le 1$. **(12 Marks)**

42. Discuss the transformation $w=\left(\dfrac{z-ic}{z+ic}\right)^2$ (*c* real) showing that the upper half of the *W*-plane corresponds to the interior of the semi circle lying to the right of imaginary axis in the *z*-plane. **(15 Marks)**

43. Use the method of contour integration to prove that $\int_0^{\pi}\dfrac{a\,d\theta}{a^2+\sin^2\theta}=\dfrac{\pi}{\sqrt{1+a^2}}\ (a>0)$.

    (15 Marks)

2002

44. Suppose that *f* and *g* are two analytic functions on the set *f* of all complex numbers with $f\left(\dfrac{1}{n}\right)=g\left(\dfrac{1}{n}\right)$ for *n=1,2,3, ...* Then show that *f(z)=g(z)* for each *z* in *f*.

    (12 Marks)

45. (i) Show that, when $0<|z-1|<2$, that function $f(z)=\frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in power of $(z-1)$ as $\frac{-1}{2(z-1)}-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$

(15 Marks)

46. Establish, by contour integration, $\int_0^{\infty}\frac{\cos(ax)}{x^2+1}dx=\frac{\pi}{2}e^{-a}$ where $a \geq 0$. **(15 Marks)**

2001

47. Prove that the Riemann zeta function ζ defined by $\zeta(z)=\sum_{n=1}^{\infty}n^{-z}$ converges for $Re z>1$ and converges uniformly for $Re z$ ³ $1+\varepsilon$ where $\varepsilon>0$ is arbitary small.

(12 Marks)

48. (i) Find the Laurent series for the function $e^{1/z}$ in $0<z<\infty$. Using this expansion, show that $\frac{1}{\pi}\int_0^{\pi}\exp(\cos\theta)\cos(\sin\theta-n\theta)\,d\theta=\frac{1}{n!}$ for n= 1,2,3,.....

(ii) Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx=\frac{\pi}{\sqrt{2}}$ **(30 Marks)**

2000

49. Show that any four given points of the complex plane can be carried by a bilinear ransformation to positions 1, -1, k and $-k$ where the value of k depeneds on the given points. **(12 Marks)**

50. Suppose f(ζ) is continuous on a circle C. Show that $\int_C\frac{f(\zeta)d\zeta}{f(\zeta-x)}$, as z varies inside of C, is differentiable under the integral sign. Find the derivative. Hence or otherwise, derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside C.

(30 Marks)

1999

51. Examine the nature of the function

$$f(z)=\frac{x^2y^5(x+iy)}{x^4+y^{10}},\ z\neq 0,\quad f(0)=0$$

In a region including the origin and hence show that Cauchy-Riemann equations are satisfied at the origin but $f(z)$ is not analytic there. **(20 Marks)**

52. For the function $\oint_C f(z)=\frac{-1}{z^2-3z+2}$ find the Laurent series for the domain

(i) $1<|z|<2$,

(ii) $|z|>2$.

Show further that $f(z)dz=0$ where C is any closed contour enclosing that points $z=1$ and $z=2$. **(20 Marks)**

53. Show that the transformation $w=\frac{2z+3}{z-4}$ transforms the circle $x^2+y^2-4x=0$ into the straight line $4u+3=0$, where $w=u+iv$. **(20 Marks)**

54. Use Residue theorem show that $\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}dx=\frac{\pi}{2}e^{-a}\sin a,\ (a>0)$ **(20 Marks)**

55. The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z|\to\infty$. If $f(2)=5$ and $f(-1)=2$ find $f(z)$. **(20 Marks)**

56. What kind of singularities the following functions have?

(i) $\frac{1}{1-e^z}$ at $z=2\pi i$

(ii) $\frac{1}{\sin z-\cos z}$ $at\ z=\frac{\pi}{4}$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ at $z=a$ and $z=\infty$.

In case (iii) above what happens when a is an integer (including $a=0$)? **(20 Marks)**

1998

57. Show that the function

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ z\neq 0 \qquad f(0)=0$$

is continuous and $C-R$ conditions are satisfied at $z=0$, but $f'(z)$ does not exist at $z=0$ **(15 Marks)**

58. Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z=-2$. Specify the region of convergence and the nature of singularity at $z=-2$ **(15 Marks)**

59. By using the integral representation of $f^n(0)$, Prove that $\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{x^n e^{xz}}{\lfloor n\, z^{n+1}} dz$

Where C is any closed contour surrounding the origin. Hence show that

$\sum_{n=0}^{\infty}\left(\frac{x^n}{\angle n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta} d\theta$ **(20 Marks)**

60. Prove that all roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$.

(15 Marks)

61. By integrating round a suitable contour show that $\int_0^{\infty} \frac{x\sin mx}{x^4+a^4} dx = \frac{\pi}{4b^2} e^{-mb} \sin mb$,

where $b = \frac{a}{\sqrt{2}}$ **(15 Marks)**

62. Using residue theorem, evaluate $\int_0^{\infty} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ **(15 Marks)**

1997

63. Prove that $u = e^x(x\cos y - y\sin y)$ is harmonic and find the analytic function whose real part is u **(15 Marks)**

64. Evaluate $\oint_c \frac{dz}{z+2}$ where C is the unit circle. Deduce that $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$

(15 Marks)

65. If $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + ... + \frac{A_n}{(Z-a)^n}$ find residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2, ..., A_n$, a and b are constant. What is the residue at infinity?

(20 Marks)

66. Find the Laurent series for the function $e^{1/z}$ in $0<|z|<\infty$. Deduce that

$\frac{1}{\pi}\int_0^{\pi} \exp(\cos\theta).\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}, (n=0,1,2...)$ **(20 Marks)**

67. Integrating e^{-z^2} along a suitable rectangular contour show that

$\int_0^{\pi} e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$ **(15 Marks)**

68. Find the function *f(z)* analytic within the unit circle, which takes the values

$\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}$, $0\le\theta\le 2\pi$ on the circle. **(15 Marks)**

1996

69. Sketchy the ellipse C described in the comples plane by $Z=A\cos\lambda t+ iB\sin\lambda t$, A>B, Where t is real variable and A,B,l are positive constants. If C is the trajectory of a particle with $z(t)$ as the position vector of the particle at time t, identify with justification
(i) The two positions where the acceleration is maximum, and
(ii) The two positions were the velocity in minimum **(20 Marks)**

70. Evaluate $\lim\limits_{z\to 0}\dfrac{1-\cos z}{\sin\left(z^2\right)}$ **(15 Marks)**

71. Show that $z = 0$ is not a branch point for the function $f(z)=\dfrac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity? **(15 Marks)**

72. Prove that every polynomial equation $a_0+a_1z+a_2z^2+....+a_nz^n=0,\ a_n\neq 0,\ n\geq 1$ has exactly n roots. **(15 Marks)**

73. By using residue theorem, evaluate $\displaystyle\int_0^\infty \frac{\log_e\left(x^2+1\right)}{x^2+1}dx$ **(15 Marks)**

74. About the singularity $z=-2$, find the Laurent expansion of $(z-3)\sin\left(\dfrac{1}{z+2}\right)$.
Specity the region of convergence and the nature of singularity at $z=-2$ **(15 Marks)**

1995

75. Let $u(x,y)=3x^2y+2x^2-y^2-2y^2$. Prove that u is a harmonic function. Find a harmonic function v such that $u+iv$ is an analytic function of z. **(15 Marks)**

76. Find the Taylor series expansion of the function $f(z)=\dfrac{z}{z^2+9}$ around $z = 0$. Find also the radius of convergence of the obtained series. **(15 Marks)**

77. Let C be the circle $|z|=2$ described counter clockwise. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z\left(z^2+1\right)}dz$$ **(20 Marks)**

78. Let a^{3}0. Evaluate the integral $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}dx$ with the aid of residues **(15 Marks)**

79. Let f be analytic in the entire complex plane. Suppose that there exist a constant $A>0$ such that $|f(z)|\leq A|z|$ for all z. Prove that there exists a complex number a such that $f(z) = az$ for all z **(15 Marks)**

80. Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ convergent at a point $z_0 \neq 0$. Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \neq 0$. Show that the series converges uniformly in the disc $\{z : |z| \le |z_1|\}$

(15 Marks)

1994

81. Suppose that z is the position vector of a particles moving on the ellipse $C : z = a\cos\omega t + ib\sin\omega t$. Where a,b, ω are positive constants, a>b and t is the time. Determine where
(i) The velocity has the greatest magnitude
(ii) The acceleration has the least magnitude. **(30 Marks)**

82. How many zeros does the polynomial $p(z)=z^4+2z^3+3z+4$ possess in (i) the first quadrant, (ii) the fourth quadrant. **(15 Marks)**

83. Test of uniform convergence in the region $|z| \le 1$ the series $\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$

(15 Marks)

84. Find Laurent series for

(i) $\frac{e^{2z}}{(z-1)^3}$ about $z=1$

(ii) $\frac{1}{z^2(z-3)^2}$ about z=3 **(30 Marks)**

85. Find the residue of $f(z)=e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

(15 Marks)

86. By means of contour integration, evaluate $\int_0^{\infty} \frac{(\log_e u)^2}{u^2+1} du$ **(15 Marks)**

1993

87. In the finite z-plane, show that the function $f(z) = \sec\left(\frac{1}{z}\right)$ has infinitely many isolated singularties in a finite intervals which includes 0. **(15 Marks)**

88. Find the orthogonal trajectories of the family of curves in the xy-plane defined by $e^{-x}(x\sin y - y\cos y) = \alpha$ where α is real function **(15 Marks)**

89. Prove that (by applying Cauchy Integral formula or otherwise)

$\int_0^{2\pi} \cos^{2n}\theta d\theta = \frac{1.3.5....(2n-1)}{2.4.6...2n} 2\pi$ where n=1,2,3... **(15 Marks)**

90. If C is the curve $y=x^2-3x^2+4x-1$ joining the points (1,1) and (2,3) find the value of $\int_c (12z^2-4iz)dz$ **(15 Marks)**

91. Prove that $\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)}$ converges absolutely for $|z|\le 1$ **(15 Marks)**

92. Evaluate $\int_0^{\infty}\frac{dx}{x^6+1}$ by choosing an appropriate contour **(15 Marks)**

1992

93. If $u=e^{-x}(x\sin y-y\cos y)$, find v such that $f(z)=u+iv$ is analytic. Also fine $f(z)$ explicitly as function of z **(15 Marks)**

94. Let $f(z)$ be analytic inside and on the circle C defined by $|z|=R$ and let $z=er^{iq}$ any point inside C. Prove that $f(er^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(\mathrm{Re}^{i\phi})}{R^2-2Rr\cos(\theta+\phi)+r^2}$ **(20 Marks)**

95. Prove that all the roots of $z^7-5z^3+12=0$ lie between the circle $|z|=1$ and $|z|=2$. **(20 Marks)**

96. Find the region of convergence of the series whose n the term is $\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$ **(20 Marks)**

97. Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(i) $|z|>3$

(ii) $1<|z|<3$

(iii) $|z|<1$ **(30 Marks)**

98. By integrating along a suitable contour evaluate $\int_0^{8}\frac{\cos mx}{x^2+1}dx$ **(20 Marks)**