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MATHEMATICS

MECHANICS AND FLUID DYNAMICS

Previous year Questions from **1992 To 2017**

Syllabus

Generalized coordinates; D' Alembert's principle and Lagrange's equations; Hamilton equations; Moment of inertia; Motion of rigid bodies in two dimensions.
Equation of continuity; Euler's equation of motion for inviscid flow; Stream-lines, path of a particle; Potential flow; Two-dimensional and axisymmetric motion; Sources and sinks, vortex motion; Navier-Stokes equation for a viscous fluid.

**** Note: Syllabus was revised in 1990's and 2001 & 2008 ****

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2017

1. Let Γ be a closed curve in xy-plane and let S denote the region bounded by the curve Γ. Let

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y)\forall(x,y)\in S.$$

If f is prescribed at each point (x,y) of S and w is prescribed on the boundary Γ of S, then prove that any solution w=w(x,y), satisfying these conditions, is unique.

[10 marks]

2. Show that the moment of interia of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length r is $\frac{1}{4}M\frac{a^2b^2}{r^2}$. Further, prove that the moment of inertia about a tangent is $\frac{5M}{4}p^2$, where p is the perpendiuclar distance from the centre of the ellipse to the tangent.

[10 marks]

3. Two uniform rods AB,AC, each of mass m and length 2a, are smoothly hinged together at A and move on a horizontal plane, At time t, the mass centre of the rods is at the point (ξ,η) referred to fixed perpendicular axes Ox, Oy in the plane and the rods make angles $\theta\pm\phi$ with Ox. Prove that the kinetic energy of the system is $m\left[\xi^2+\eta^2+\left(\frac{1}{3}+\sin^2\phi\right)a^2\theta^2+\left(\frac{1}{3}+\cos^2\phi\right)a^2\phi^2\right]$

Also derive Lagrange's equations of motion for the system if an external force with components [X,Y] along the axes acts at A. **[20 marks]**

4. A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D ad d. If V and v be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vertex of the cone, then prove that

$$\frac{v}{V}=\frac{D^2}{d^2}e^{\left(v^2-V^2\right)/2K}$$

Where K is the pressure divided by the density and is constant. **[15 marks]**

5. If the velocity of an incompressible fluid at the point (x,y,z) is given by $\left(\frac{3xz}{r^5},\frac{3yz}{r^5},\frac{3z^2-r^2}{r^5}\right), r^2=x^2+y^2+z^2$, then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the streamlines **[15 marks]**

2016

6. Consider a single free particle of mass m, moving in space under no forces. If the particle starts from the origin at t=0 and reaches the position (x,y,z) at time τ, find the Hamilton's characteristic function S as a function of x,y,z, τ. **[10 marks]**

7. A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity U$\vec{i}$. Show that the velocity potential ϕ at any point P of the stream is $\frac{\mathrm{m}}{\mathrm{r}}-\mathrm{Ur}\cos\theta$, where OP=r and θ is the angle which $\overline{OP}$ makes with the direction $\vec{i}$. Find the differential equation of the streamlines and show that they lie on the surfaces

$Ur^2\sin^2\theta - 2m\cos\theta = \text{constant.}$ **[15 marks]**

8. The space between two concentric spherical shells of radii a,b (a<b) is filled with a liquid of density p. If the shells are set in motion, the inner one with velocity U in the x-direction and the outer one with velocity V in the y-direction, then show that the initial motion of the liquid is given by velocity potential

$$\phi = \frac{\left\{a^3U\left(1+\frac{1}{2}b^3r^{-3}\right)x - b^3v\left(1+\frac{1}{2}a^3r^{-3}\right)y\right\}}{\left(b^3-a^3\right)}$$

where $r^2 = x^2 + y^2 + z^2$, the coordinates being rectangular. Evaluate the velocity at any point of the liquid. **[20 marks]**

9. A hoop with radius r is roling without slipping, down an inclined plane of length l and with angle of inclination ϕ; Assign appropriate generalized coordinates to the system. Determine the constraints, if any. Wrtie down the Lagrangian equations for the system. Hence or otherwise determine the velocity of the hoop at the bottom of the inclined plane. **[15 marks]**

2015

10. Consider a uniform flow U_0 in the positive x-direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. **[10 marks]**

11. Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass m about the OZ-axis. **[10 marks]**

12. Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion. **[15 marks]**

13. A Hamiltonian of a system with one degree of freedom has the form

$$H = \frac{p2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2}q^2e^{-\alpha t(\alpha + be^{-\alpha t})} + \frac{k}{2}q^2$$ where α ,b, k are constants, q is the generalized coordinate and p is the corresponding generalized momentum.

(i) Find a Lagrangian corresponding to this Hamiltonian.

(ii) Find an equivalent Lagrangian that is not explicitly dependent on time. **[10+10=20 marks]**

14. In an axisymmetric motion, show that stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisfied by the stream function if the flow is irrotational. **[20 marks]**

2014

15. Find the equation of motion of a compound pendulum using Hamilton's equations. **[10 marks]**

16. Find Navier-Stokes equation for a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates. **[20 marks]**

2013

17. Four Solid spheres A,B,C and D, each of mass m and radius a, are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about a diagonal of the

sqaure. **[10 marks]**

18. Two equal rods AB an BC, each of length l. smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$. **[10 marks]**

19. If fluid fills the region of space on the positive side of the x-axis, which is a rigid boundary and if there be a source m at the point (0,a) and an equal sink at (0,b) and if the pressure on the negative side be the same as the pressure at infinity, show that the resultant pressure on the boundary is $\frac{\pi\rho \mathrm{m}^2 (\mathrm{a}-\mathrm{b})^2}{\{2\mathrm{ab}(\mathrm{a}+\mathrm{b})\}}$ where ρ is the density of the fluid. **[15marks]**

20. If n rectilinear vortices of the same strength K are symmetrically arranged as generators of circular cylinder of radius a in an infinite liquid, prove that the vortices will move the cylinder uniformly in time $\frac{8\pi^2 \mathrm{a}^3}{(\mathrm{n}-1)\mathrm{K}}$. Find the velocity at any point of the liquid. **[10 marks]**

2012

22. Obtain the equations governing the motion of a spherical pendulum. **[12 marks]**

23. A rigid sphere of radius **a** is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the disturbed stream. **[12 marks]**

24. A pendulum consists of a rod of length 2a and mass m; to one end of which a spherical bob of radius a/3 and mass 15m is attached. Find the moment of inertia of the pendulum:
(i) about an axis through the other end of the rod and at right angles to the rod.
(ii) about a parallel axis through the centre of mass of the pendulum.
[Given: The centre of mass of the pendulum is a/12 above the centre of the sphere].
[15 marks]

25. Show that $\phi = \mathrm{x} f(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.
[30 marks]

2011

26. Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendiuclar to the axis.
[12 marks]

27. The ends of a heavy rod of length 2a are rigidly attached to two light rings which can respectively slide on the thin smooth fixed horizontal and vertical wires O_X and O_y. The rod starts at an angle α to the horizon with an angular velocity $\sqrt{[3\mathrm{g}(1-\sin\alpha)/2\mathrm{a}]}$ and moves downwards. Show that it will strike the horizontal wire at the end of time $2\sqrt{a/(3g)}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right]$
[30 marks]

28. An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength K but they are alternately of opposite signs. Find the Complex function that

determines the velocity potential and the stream function. **[30 marks]**

2010

29. A uniform lamina is bounded by a parabolic arc of latus rectum 4a and a double ordinate at a distance b from the vertex.

If $b=\frac{a}{3}\left(7+4\sqrt{7}\right)$, show that two of the principal axis at the end of a latus rectum are the tangent and normal there. **[30 marks]**

30. In an incompressible fluid the vorticity at every point is constant in magnitude and direction; show that the components of velocity u,v,w are solutions of Laplace's equation. **[30 marks]**

31. A sphere of radius a and mass m rolles down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations. **[30 marks]**

32. When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation.

$\left(r^2\sin^2\theta-b^2\right)\left(r^2-a^2\right)^2=4a^2b^2r^2\sin^2\theta$, θ being measured from the line through the centre perpendicular to the joint of the vortices. **[30 marks]**

2009

33. The flat surface of a hemisphere of radius r is cemeted to one flat surface of a cylinder of the same radius and of the same material. If the length of the cylinder be l and the total mass be m, show that the moment of inertia of the combination about the axis of the cylinder is given by:

$\mathrm{m}r^2\left(\frac{l}{2}+\frac{4}{15}r\right)/\left(l+\frac{2r}{3}\right)$. **[12 marks]**

34. Two sources, each of strength m are places at the points (–a,0), (a,0) and a sink of strength 2 m is at the origin. Show that the stream lines are the curves:

$\left(x^2+y^2\right)^2=a^2\left(x^2-y^2+\lambda xy\right)$

where λ is a variable parameter.

Show also that the fluid speed at any point is $(2ma^2)/(r_1r_2r_3)$, where r_1,r_2 and r_3 are the distances of the points from the sources and the sink. **[12 marks]**

34. A perfectly rough sphere of mass m and radius b, rests on the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m' and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length

$(a-b)\dfrac{4m'+\frac{7}{5}m}{m+m'\left(2-\frac{a}{b}\right)}$. **[30 Marks]**

35. An infinite mass of fluid is acted on by a force $\frac{\mu}{r^{3/2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r=C$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2}.C^{5/4}$. **[30 Marks]**

2008

35. A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board? Explain. What happens if the mass of the board and boy are equal? **[12 marks]**

36. If the velocity potential of a fluid is $\phi = \frac{1}{r^3} z . \tan^{-1}\left(\frac{y}{x}\right), r^2 = x^2 + y^2 + z^2$ then show that the stream lines on the surfaces $x^2+y^2+z^2=c\,(x^2+y^2)^{2/3}$, c being a constant. **[12 marks]**

37. A uniform rod of mass 3m and length $2l$ has its middle point fixed and a mass m is attached to one of its extremily. The rod, when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity $\sqrt{\frac{2g}{l}}$. Show that the heavy end of the rod will fall till the inclination of the rod of the vertical is $\cos^{-1}\left(\sqrt{2}-1\right)$. **[30 marks]**

38. Let the fluid fills the region $x \geq 0$ (right half of 2d plane). Let a source α be at $(0,y_1)$ and equal sink at $(0,y_2)$, $y_1>y_2$. Let the pressure be same as pressure at infinity i.e. P_o. Show that the resultant pressure on the boundary (y-axis) is $\pi\rho\alpha^2\left(y_1-y_2\right)^2/2y_1y_2\left(y_1+y_2\right)$, ρ being the density of the fluid. **[30 marks]**

2007

39. Consider a system with two degrees of freedom for which
$V = q_1^2 + 3q_1 q_2 + 4q_2^2$.
Find the equilibrium position and determine if the equilibrium is stable. **[12 marks]**

40. Show that $\left(\frac{x^2}{a^2}\right)\cos^2 t + \left(\frac{y^2}{b^2}\right)\sec^2 t = 1$ is a possible form for the boundary surface of a liquid. **[12 marks]**

41. A point mass m is placed on a frictionless plane that is tangent to the Earth's surface. Determine Hamilton's equations by taking x or θ as the generalized coordinate. **[30 marks]**

42. A thin plate of very large area is placed in a gap of height h with oils of viscosities μ' and μ'' on the two sides of the plate. The plate is pulled at a constant velocity v. Calculate the position of the plate so that
(i) the shear force on the two sides of the plate is equal.
(ii) The force required to drag the plate is minimum. [End effects are neglected.] **[30 marks]**

2006

43. Given points A(0,0) and B(x_0,y_0) not in the same vertical, it is required to find a curve in the x–y plane joining A to B so that a particle starting from rest will traverse from A to B along this curve without friction in the shortest possible time. If y=y(x) is the required curve find the function f(x,y,z) such that the equation of motion can be written as $\frac{dx}{dt} = f\left(x, y(x), y'(x)\right)$. **[12 marks]**

44. A steady invisicid incompressible flow has a velocity field u=fx, v=–fy, w=0

where f is a constant. Derive an expression for the pressure field p{x,y,z} if the pressure p{0,0,0}=P_0 and $\vec{g} = -g\vec{i}z$. **[12 marks]**

45. A particle of mass m is constrained to move on the surface of a cylinder. The particle is subjected to a force directed towards the origin and proportional to the distance of the particle from the origin. Construct the Hamiltonian and Hamilton's equations of motion. **[30 marks]**

46. Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if P be the pressure at the outer surface, the initial pressure at any point on the liquid, distant r from the centre is $P\frac{\log r - \log b}{\log a - \log b}$ **[30 marks]**

2005

47. A rectangular plaste swings in a vertical plane about one of its corners. If its period is one second, find the length of its diagonal. **[12 marks]**

48. Prove that the necessary and sufficient condition for vortex lines ans stream lines to be at right angles to each other is that **[12 marks]**

$$u = \mu\frac{\partial\varphi}{\partial x}, v = \mu\frac{\partial\varphi}{\partial y}, w = \mu\frac{\partial\varphi}{\partial z},$$

Where μ and φ are functions of x,y,z and t.

49. A plank of mass M, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time.

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

Where a is the length of the plank. **[30 marks]**

50. State the conditions under which Euler's equations of motion can be integrated. Show that

$$-\frac{\partial\varphi}{\partial t} + \frac{1}{2}q^2 + V + \int\frac{dp}{\rho} = F(t)$$

Where the symbols have their usual meaning. **[30 marks]**

2004

51. A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$. Which is assumed frictionless. Obtain the equation of motion. Show that it will describe a horizontal circle in the plane z= h, provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2g/a}$. **[12 marks]**

52. In an incompressible fluid, the vorticity at every point is constant, in magnitude and direction. Do the velocity components satisfy the Laplace Equation? Justify. **[12 marks]**

53. Derive the Hamilton equations of motion from the principle of least action and obtain the same for a particle of mass m moving in a force field of potential V.

Write these equations in spherical coordinates (r, θ, ϕ). **[30 marks]**

54. The space between two infinitely long coaxial cylinders of radii a and b (b>a) respectively is filled by a homogeneous fluid, of density ρ. The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder is.

$$\pi\rho a^2 l \frac{b^2+a^2}{b^2-a^2} v.$$ **[30 marks]**

2003

55. A solid body of density ρ is in the shpae of the solid formed by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line. Show that its moment of inertia about the staright line through the pole and perpendicular to the initital line is $\frac{352}{105}\pi\rho a^2$ **[12 marks]**

56. For an incompressible homogeneous fluid at the point (x,y,z) the velocity distribution is given by $u = -\frac{c^2 y}{r^2}$, $v = \frac{c^2 x}{r^2}$, w=0, where r denotes the distance from z-axis. Show that it is a possible motion and determine the surface which is orthogonal to stream lines. **[12 marks]**

57. A fine circular tube, radius c, lies on a smooth horizontal plane, and contains two equal particles connected by an elastic string in the tube, the natural length of which is equal to half the circumference. The particles are in contact and fastened together, the string being stretched round the tube. If the particles become disunited, prove that the velocity of the tube when the string has regained its natural length is

$\left\{\frac{2\pi\lambda m c}{M(M+2m)}\right\}^{1/2}$ when M, m are the masses of the tube and each particle respectively, and λ is the modulus of elasticity. **[30 marks]**

58. Two sources, each of strength m are placed at the points (–a,0) and (a,0) and a sink of strength 2 m is placed at the origin. Show that the stream lines are the curves $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$, where λ is a variable parameter. Also show that the fluid speed at any point is $\frac{2ma^2}{r_1 r_2 r_3}$, where r_1, r_2 and r_3 are respectively the distances of the point from the sources and sink. **[15 marks]**

59. An infinite mass of fluid is acted upon by a force $u\, r^{-3/2}$ per unit mass directed to the origin. If inititally the fluid is at rest and there is a cavity in the form of a sphere r = c in it; show that the cavity will be filled up after an interval of time $\{2/5\mu\}^{1/2} c^{5/4}$. **[15 marks]**

2002

60. Find the moment of inertia of a circular wire about
(i) a diameter; and
(ii) a line through the centre and perpendicular to its plane. **[12 marks]**

61. Show that the velocity potential

$\phi = 1/2a(x^2+y^2-2z^2)$ satisfies the Laplace equation, and determine the stream lines.

[12 marks]

62. A thin circular disc of mass M and radius a can turn freely about a thin axis OA, which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity w about its end A. Show that the inclination θ to be vertical of the radius of the disc through O is $\cos^{-1}\left(g/aw^2\right)$ unless $w^2 < g/a$ and then θ is zero.

[30 marks]

63. Show that $u = \frac{-2xyz}{\left(x^2+y^2\right)^2}, v = \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, w = \frac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrotataional?

[15 marks]

64. Prove that

$$\left(v\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial\left(x, y\right)}$$

where v is the kinematic viscosity of the fluid and ψ is the stream function for a two-dimensional motion of a viscous fluid. **[15 marks]**

2001

65. Determine the moment of inertia of a uniform hemisphere about its axis of symmetry and about an axis perpendicular to the axis of symmetry and through centre of the base. **[6 marks]**

66. If the velocity distribution of an incompressible fluid at the point (x,y,z) is given by

$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{kz^2-r^2}{r^5}\right)$ then determine the parameter k such that it is possible motion. Hence find its velocity potential. **[12 marks]**

67. Find the equation of motion for a particle of mass m which is constrained to move on the surface of a cone of semi-vertical angle α and which is subjected to a gravitational force. **[30 marks]**

68. Show that the velocity distribution in axial flow of viscous incompressible fluid along a pipe of annular cross-section, radii $r_1 < r_2$, is given by

$$\omega(r) = \frac{1}{4\mu}\frac{dp}{dz}\left\{r^2 - r_1^2 + \frac{r_2^2 - r_1^2}{\log\left(r_2/r_1\right)}\log\left(\frac{r}{r_2}\right)\right\}$$

[30 marks]

2000

69. Find the moment of inertia of an ellipte area about a line CP inlined at () to the major axis and about a tangent parallel to CP, where C is the centre of the ellipse. **[15 marks]**

70. Dtermine the stream lines and the path lines of the particle when the components of the velocity field are given by $u = \frac{x}{1+t}, v = \frac{y}{2+t}$, and $w = \frac{z}{3+t}$. Also state the condition for which the stream lines are identical with the path lines. **[20 marks]**

71. A plank of mass **M**, is inititally at rest along a line of greatest sloope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$ where a is the length of the plank. **[15 marks]**

72. Define irrotational and rotational flows giving an example for each.
Show that

$$u = \frac{-2xyz}{\left(x^2+y^2\right)^2}, v = \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, w = \frac{y}{\left(x^2+y^2\right)},$$

are the velocity components of a possible liquid motion. Examine this for irrotational motion.

[15 marks]

73. An explosion in a factory manufacturing explosives can occur because of
(i) leakge of electricity
(ii) defects in machinery
(iii) carelessness of workers or
(iv) sabotage
The probability that there is leakge of electricity is 0.20, the machinery is defective is 0.30, the workers are caresless is 0.40, there is sabotage is 0.10. The engineers feel that an explosion can occur with probability
(i) 0.25 because of leakage of electricity
(ii) 0.20 because of defects in machinery
(iii) 0.50 because of carelessness of workers and
(iv) 0.75 beacuse of sabotage.
Determine the most likely cause of explosion. **[15 marks]**

1999

74. A particle of given mass m moves in space with the Lagrangian
$L = 1/2\, m\left(x^2+y^2+z^2\right) - V + xA + yB + zC$. where V, A, B, C are given functions of x,y,z. Show that the equations of motion are $mx = -\frac{\partial V}{\partial x} + y\left[\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right] - z\left[\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right]$ and two similar equations for y and z. Find also the Hamiltonian H in term of generalized momenta.

[20 marks]

75. A wheel consits of a thin rim of mass M and n evently placed spokes each of mass m, which may be considered as thin rods terminating at the centre of the wheel. If the wheel is rolling with linear velocity v, express its kinetic energy in terms of M, m, n, v. With what acceleration will it roll down a rough inclined plane of inclination α? **[20 marks]**

76. Find the moment of inertia of a solid hemisphere about a diameter of its plane base.
A solid hemisphere is held with its base against a smooth vertical wall and its lowest point on a smooth floor. The hemisphere is released. Find the initital reactions on the wall and the floor.

[20 marks]

77. Derive the equation of continuity for a fluid in which there are no sources or sinks. Liquid flows through a pipe whose surface is the surface of revolution of the curve $y=a+kx^2/a$ about the x-axis ($-a \le x \le a$). If the liquid enters at the end $x=-a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x=-a$ to $x=+a$ is

$\left\{\frac{2a}{V(1+k)^2}\right\}\left(1+\frac{2}{3}k+\frac{1}{5}k^2\right)$(Assume that k is so small that the flow remains appreciably one-dimensional throughout). **[20 marks]**

78. A spherical globule of gas initially of radius R_0 and at pressure P_0 exapnds in an infinite mass of water of density ρ in which the pressure at infinity is zero. The gas is initially at rest and its pressure p and volume v are governed by the equation $pv^{4/3}$ = constant. Prove that the gas doubles its radius

in time $\frac{28R_0}{15}\left(\frac{2\rho}{P_0}\right)^{1/2}$. **[20 marks]**

79. Two sources each of strength m are placed at point (–a,0) and (a,0) and a sink of strength 2 m is placed at the origin. Show that the stream lines are the curves.

$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2+\lambda xy\right)$ where λ is a variable parameter. **[20 marks]**

80. A prinitng machine can print n "letter", say $\alpha_1, \alpha_2, ..., \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that there exists a constant probability p of printing the correct letter and also assume independence. One of the n impulses, chosen at random, was fed into the machine twice and both times the letter α_1 was printed. Compute the probability that the impulse chosen was meant to print α_1. **[20 marks]**

81. A physical quantity is measured many times for accuracy. Each measurement is subject to a random error. It is judged reasonable to assume that it is uniformly distributed between –1 abnd +1 in a conveniently chosen unit. How many measurements should be taken in order that the probability will exceed 0.95 that the average will differ from true value by at most 0.2? **[20 marks]**

1998

82. Two particles in a plane are connected by a rod of constant length and are constrained to move in such a manner that the velocity of the middle of the rod is in the direction of the rod. Write down the equations of the constraints. Is the system holonomic or non-holonomic? Give reason for your answer. **[20 marks]**

83. Using Lagrannge equations, obtain the differential equations of motion of a free particle in spherical polar coordinates. **[20 marks]**

84. A rod of length 2a is suspended by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity ω, and their inclinations to the vertical be α and β respectively. show that $\omega^2 = \frac{3g\tan\beta}{3l\sin\alpha + 4a\sin\beta}$ **[20 marks]**

85. A particle of mass m is fixed to a point P of the rim of a uniform circular disc of centre , mass m and radius a. The disc is held, with its plane vertical its lowest point in contact with a perfectly rough horizontal table and with OP inclined at 60^o to the upward vertical and is then released. If the subsequent motion continues in the same vertical plane, show that, when OP makes and angle θ with the upward vertical $a(7+4\cos\theta)\theta^2 = 2g(1-2\cos\theta)$ **[20 marks]**

Show also that when OP is first horizontal, the acceleration of σ is $\frac{18}{49}$ g.

86. Three equal uniform rods AB,BC,CD each of mass m and length 2a, are at rest in a stright line smoothly jointed at B and C. A blow J is given to the meddle rod at a distance x from its centre σ in a direction perpendiuclar to it; show that the initial velocity of σ is $\frac{2J}{3m}$, and that the initital angular velocities of the rods are: $\frac{5a+9x}{10ma^2}J, \frac{6x}{5ma^2}J, \frac{5a-9x}{10ma^2}J,$ **[20 marks]**

87. Show that a fluid of constant density can have a velocity $\vec{q}$ given by:

$\vec{q}=\left[\frac{-2xyz}{\left(x^2+y^2\right)^2},\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2},\frac{y}{x^2+y^2}\right]$ Determine if the fluid motion is irrotational. **[20 marks]**

88. Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the steam, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$\frac{v}{V}=\left(\frac{D}{d}\right)^2 e^{\frac{v^2-V^2}{2k}}$ where k is the pressure divided by the density, and supposed constant.

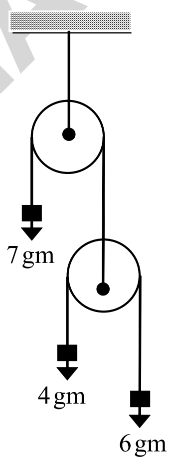
[20 marks]

89. Between two fixed boundaries $\theta=\frac{\pi}{4}$ and $\theta=\frac{-\pi}{4}$ there is a two-dimensional liquid motion due to a source of strength m at the point (r=a, θ=0), and an equal sink at the point (r=b, θ=0) show that the stream function is

$-m\tan^{-1}\frac{r^4\left(a^4+b^4\right)\cos 4\theta}{r^6-r^4\left(a^4+b^4\right)\cos 4\theta+a^4b^4}$ **[20 marks]**

1997

90. A pulley system is given as shown in the diagram. Discuss the motion of the system using Lagrange's method when the pulley wheels have negligible masses and moments of inertia and their wheels are frictionless. **[20 marks]**



91. Show that $\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t=1$ is possible form for the bounding surface of a liquid, and find an expression for the normal velocity. **[20 marks]**

92. A stream in a horizontal pipe, after passing a contraction in the pipe, at which its cross-sectional area is A, is delivered at the atmospheric pressure at a place where the cross sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into pipe from a reservoir at a depth $\frac{s^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)$ below the pipe where s is the delivery per second. **[20 marks]**

93. Using the method of images prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$, the solution is usual notations, $\phi + i\psi = -m\log\left[\left(z^3 - z_0^3\right)\left(z^3 - z_0^{'3}\right)\right]$ where $z_0 = x_0 + i\,y_0$ and $z_0 = x_0 - i\,y_0$. **[20 marks]**

1996

94. A perfectly rough circular hoop of diameter 24 cm rolls on a horizontal floor with velocity V cm/sec towards an inelastic step of height 4 cm, the plane of the hoop being vertical and perpendicular to the edge of the step. Prove that hoop can mount the step without losing contact at any stage if

$2.4\sqrt{2g} > V > 2.4\sqrt{g}$ **[20 marks]**

95. A homogeneous sphere rolls down an imperfectly rough fixed sphere, starting from rest at the highest point. If the sphere separtes when the line joining their centres makes an angle 30° with the vertical, show that the coefficient of friction μ satisfies the following equaiton:

$$e^{\mu\pi/3} = \frac{3\sqrt{3} + 6m}{4\left(1 - 2\mu^2\right)}$$ **[20 marks]**

96. Find the stream function of two-dimensional motion due to two equal sources and an equal sink situated midway between them.
In a region bounded by a fixed quadrantal arc and its radii deduce the motion due to a source and an equal sink situated, at the end of one of the bounding radii. Show that the stream line leaving either end at an angle $\pi/6$ with the radius is

$r^2\sin\left(\frac{\pi}{6} + \theta\right) = a^2\sin\left(\frac{\pi}{6} - \theta\right)$, where a is the radius of the quadrant. **[20 marks]**

1995

97. (a) How do you characterzie
(i) the simplest dynamical system?
(ii) the most general dynamical system?
Show that the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial q_k} - \frac{\partial T}{\partial q_k} = Q_k, (k = 1, 2, ..., n)$$

correspond to a nonconservative but sceleronomic and holonomic system with n degrees of freedom, whre q_J, q_k, Q_k are respectively the generalized coordinates, the generalized velocties and the generalized forces. **[20 marks]**

98. A solid uniform sphere has a light rod rigidly attached to it which passes through the centre. The rod is so joined to a fixed vertical axis that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve contantly with uniform angular velocity, show that θ^2 is of the form

$$n^2\left(\cos\theta - \cos\beta\right)\left(\cos\alpha - \cos\theta\right)$$

whre n, α, β are certain constants. **[20 marks]**

99. A uniform rod of length 20 cms which has one end attached to a fixed point by a light inextensible string of length $4\frac{1}{6}$ cms, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time and the periods of principal oscillations.

[20 marks]

100. A carriage is placed on an inclined plane making an angle α with the horizon and rolls down without slipping between the wheels and the plane. The floor of the carriage is parallel to the plane and a perfectly rough ball is placed freely on it. Show that the acceleration of the carriage down the plane is

$$\frac{14M+4m'+14m}{14M+4m'+21m} g \sin\alpha$$

whre M is the mass of the carriage excluding the wheels, m the sum of the masses of the wheels which are uniform circular discs and M' that of the ball which is a homogeneous solid sphere (the frictioni between the wheels and the axes is neglected). Show that for the motion to be possible, the coeffiecient of friction between the wheels and the plane must exceed the constant

$$\frac{7(M+m)+2M'}{14M+21m+4M'}\tan\alpha$$ **[30 marks]**

101. A sphere of radius a is projected up an inclined plane with velocity V and angular velocity Ω in the sense which could cause it to roll up; if V > a Ω and the coeffiecient of friction greater than $\frac{2}{7}$ $\tan\alpha$, show that the sphere will cease to ascend at the end of a time

$$\frac{5V+2a\Omega}{5g\sin\alpha}$$

where α is the inclination of the plane. **[30 marks]**

102. Determine the restriction on f_1, f_2, f_3 if

$$f_1(t)\frac{x^2}{a^2}+f_2(t)\frac{y^2}{b^2}+f_3(t)\frac{z^2}{c^2}=1$$

is a possible boundary surface of a liquid. **[30 marks]**

103. If the fluid fill the region of spaces on the positive side of x-axis, which is a rigid boundary and if there be a source +m at the point (0,a) and an equal sink at (0,b) and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is

$$\pi\rho m^2 \frac{(a-b)^2}{ab(a+b)}$$

where ρ is the density of the fluid. **[20 marks]**

104. If a,b,c,d,e,f are arbitrary constants, what type of fluid motion does the velocity (a+by-cz, d-bx+ez, f+cx-ey) represent? **[20 marks]**

1994

104. The particle velocity for a fluid motion referred to rectangular axes is given by

$\left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, 0, A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$ whre A, a are constants. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube

$-a \le x \le a, \quad 0 \le z \le 2a$. Also find the pressure associated with this velocity field. **[20 marks]**

105. Determine the stremlines and the path lines of the particles when the velocity field is given by

$\left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right)$ **[20 marks]**

106. Between the fixed boundaries $\theta = \frac{\pi}{4}$ are $\theta = -\frac{\pi}{4}$, there is a two dimensional liquid motion due to a source of strength m at the point r =a, $\theta = 0$ and an equal sink at the point r = b, q =0. Use the method of images to show that the stream function is $-m\tan^{-1}\left\{\frac{r^4\left(a^4-b^4\right)\sin 4\theta}{r^3-r^4\left(a^4+b^4\right)\cos 4\theta+a^4b^4}\right\}$

Show also that the velocity at (r,θ) is

$$\frac{4m\left(a^4-b^4\right)r^3}{\left(r^8-2a^4r^4\cos 4\theta+a^8\right)^{1/2}\left(r^8-2b^4r^4\cos 4\theta+b^8\right)^{1/2}}$$ **[20 marks]**

1993

107. Find
(i) the Lagrangian
(ii) the equations of motion
for the following system:
A particle is constrained to move in a plane under the influence of an attraction towards the orgian proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction.
[30 marks]

108. A heavy uniform rod rotating in a vertical plane falls and strikes a smooth inelastic horizontal plane. Find the impulse. **[20 marks]**

109. The door of a railway carriage has its hinges, supposed smooth, towards the engine, which starts with an acceleration f. Prove that the door closes in time.

$$\left(\frac{a^2+K^2}{2\,af}\right)^{1/2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$$

with an angular velocity $\sqrt{\frac{2af}{a^2+K^2}}$ where 2a is the breadth of the door and K its radius of gyration about a vertical axis through G, the centre of mass **[30 marks]**

110. A solid homogenous sphere is resting on the top of another fixed sphere and rolls down it. Write down the equations of motion and find the friction. When does the upper sphere leave the lower sphere if
(i) both the spheres are smooth
(ii) the upper sphere is sufficiently rough so as not to slip. **[30 marks]**

111. Show that

$$u=\frac{-2xyz}{\left(x^2+y^2\right)},\ v=\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2},\ w=\frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational?
(b) Stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V}=\frac{D2}{d^2}e\left(v^2-V^2\right)/2K$$

where K is the pressure divided by the density and supposed constant. **[20 marks]**

112. Prove that for liquid circulating irrotationally in part of the plane between two non intersecting circles, the curves of constant velocity are Cassinis' ovals. **[20 marks]**

1992

113. Classify each of the following dynamical systems according as they are
(i) sclernomic or rheonomic
(ii) holonomic or nonholonomic
(iii) conservative or non conservative
[I] a horizontal cylinder of radius a rolling inside a perfectly rough hollow horizontal cylinder of raiuds b>a.
[II] a particle constrained to move along a line undr the influence of a force which is inversely proportional to the square of its distance from a fixed point and a dampipng force proportional to the sqaure of the instantaneous speed.
[III] a particle moving on a very long frictionless wire which rotates with constant angular speed about a horizontal axis. **[10 marks]**

114. When the Lagrangian function has the form

$$L=q_k q_k-\sqrt{\left(1-q_k^2\right)}$$

show that the generalized acceleration is zero. **[20 marks]**

115. The ends of a uniform rod AB of length 2a cos15° and weight W are constrained to slide on a smooth circular wire of radius a fixed with its plane vertical. The end A is connected by an elastic string of natural length a and modulus of elasticity W/2 to the highest point of the wire. If θ is the angle which the perpendicular bisector of the rod makes with the downward vertical, show that the potential energy V is given by

verify $V=-\frac{W_a}{2}\left\{\cos\left(\theta-75^0\right)+2\cos 1/2\left(\theta+75^0\right)\right\}+\text{constant}$

Verif that $\theta=25^0$ defines a position of equilibrium and inverstigate is stability. **[30 marks]**

116. A uniform rod of length 2a which has one end attached to a fixed point by alight inextensible string of length 5a/12 is performing small oscillations in a vertical plane about its position of equilibrium. Find the position at any time and show that the periods of its principal oscillations are

$2\pi\sqrt{\frac{5a}{3g}}$ and $\pi\sqrt{\frac{a}{3g}}$ **[30 marks]**

117. Show that the variable ellipsoid

$$\frac{x^2}{a^2k^2t^4}+kt^2\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=1$$

is a possible form for the boundary surface of a liquid motion at any time t. **[30 marks]**

118. Find the lines of flow in the 2-dimensional fluid motion given by

$\phi + i\psi = -1/2^n (x+iy)^2 e^{2int}$ **[20 marks]**

119. A source of strength *m* and a vortex of strength k are placed at the origin of the 2-dimensional motion of unbounded liquid. Prove that the pressure at infinity exceeds that pressure at distance r from the origin by $\frac{1}{2} - \frac{(m^2+k^2)}{r^2} p$ **[20 marks]**